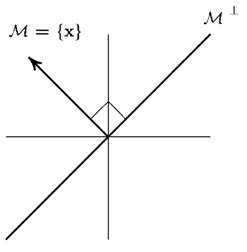
13. Ortogonalna dekompozicija

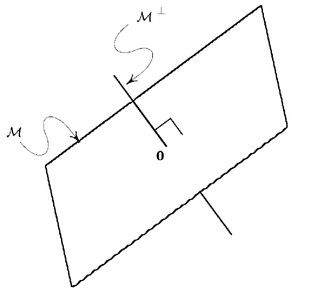
(13.01) Ortogonalni komplement

Za podskup M unitarnog prostora $\mathcal{V}$, ortogonalni komplement $\mathcal{M}^{\perp}$ (čitaj ”M nor”) od $\mathcal{M}$ je definisan kao skup svih vektora iz $\mathcal{V}$ koji su ortogonalni na svaki vektor iz $\mathcal{M}$. To jest

$$\mathcal{M}^{\perp} = \{x \in \mathcal{V} \,|\, \langle m, x\rangle = 0 \text{ za svaki } m \in \mathcal{M}\}.$$

$\diamond$





(13.02) Ortogonalno komplementarni podprostor

Ako je $\mathcal{M}$ podprostor konačno dimenzionalnog unitarnog podprostora $\mathcal{V}$, tada je

$$\mathcal{V} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

Štaviše, ako je $\mathcal{N}$ podprostor takav da $\mathcal{V} = \mathcal{M} \oplus \mathcal{N}$ i $\mathcal{N} \perp \mathcal{M}$ (svaki vektor u $\mathcal{N}$ je ortogonalan na svaki vektor u $\mathcal{M}$), tada

$$\mathcal{N} = \mathcal{M}^{\perp}.$$

$\diamond$

(13.03) Nor operator

Ako je $\mathcal{M}$ podprostor konačno dimenzionalnog unitarnog podprostora dimenzije n, tada su sljedeće tvrdnje tačne

- $dim(\mathcal{M}^{\perp}) = n - dim(\mathcal{M})$.
- $\mathcal{M}^{\perp^{\perp}} = \mathcal{M}$ $\diamond$

(13.03) Teorema ortogonalne dekompozicije

Za svaku matricu $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$

$$im(A)^{\perp} = ker(A^{\top}) \quad \text{i} \quad ker(A)^{\perp} = im(A^{\top}).$$

Ako iskoristimo prvu osobinu iz 13.02 ovo znači da svaka matrica $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ pravi ortogonalnu dekompoziciju od $\mathbb{R}^m$ i $\mathbb{R}^n$ u smislu da

$$\mathbb{R}^m = im(A) \oplus im(A)^{\perp} = im(A) \oplus ker(A^{\top}),$$

i

$$\mathbb{R}^n = ker(A) \oplus ker(A)^{\perp} = ker(A) \oplus im(A^{\top}),$$

$\diamond$

(13.04) URV faktorizacija

Za svaku matricu $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ ranga r postoje ortogonalne matrice $U_{m\times m}$ i $V_{n\times n}$ i nesingularna matrica $C_{r\times r}$ takve da

$$A = URV^{\top} = U \begin{pmatrix} C_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_{m\times n} V^{\top}.$$

- Prvih r kolona u U čine ortonormiranu bazu za $im(A)$.
- Zadnjih $m - r$ kolona u U čine ortonormiranu bazu za $ker(A^\top)$.
- Prvih r kolona u V su ortonormirana baza za $im(A^\top)$.
- Zadnjih $n - r$ kolona u V su ortonormiranu bazu za $ker(A)$.

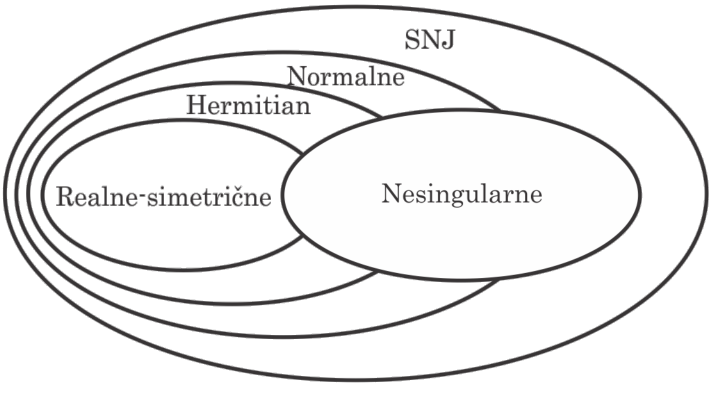
Svaka različita familija ortonormiranih baza za četri fundamentalna podprostora od A proizvodi različitu URV faktorizaciju od A. U kompleksnom slučaju, $(\star)^\top$ mjenjamo sa $(\star)^*$ i "ortogonalno" mijenjamo sa "unitarno". $\diamond$

(13.05) Slika normalna na jezgro

Za $rang(A_{n\times n}) = r$, sljedeće tvrdnje su ekvivalentne:

- $im(A) \perp ker(A)$,
- $im(A) = im(A^\top)$,
- $ker(A) = ker(A^\top)$,
- $A = U \begin{pmatrix} C_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} U^\top$

gdje je U ortogonalna a C nesingularna matrica. Ovakve matrice ćemo zvati <u>*SNJ matrice*</u> skraćenica od "slika normalna na jezgro". Neki autori ih nazivaju rang-simetrične ili EP ili RPN matrice. Nesingularne matrice su trivijalne SNJ matrice zato što je jezgro nula. U kompleksnom slučaju, $(\star)^\top$ mjenjamo sa $(\star)^*$ i "ortogonalno" mijenjamo sa "unitarno". $\diamond$



Provjeriti teoremu ortogonalne dekompozicije za matricu

$$A=\begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}.$$

Rj: Prisjetimo se

Teorema ortogonalne dekompozicije

Za svaku matricu $A \in Mat_{m\times n}(\mathbb{R})$

$$im(A)^{\perp} = ker(A^T) \quad i \quad ker(A)^{\perp} = im(A^T)$$

Kako za svaki podprostor $\mathcal{M} \subseteq \mathbb{R}^k$ vrijedi $\mathbb{R}^k = \mathcal{M} \oplus \mathcal{M}^{\perp}$, to znači da svaka matrica $A \in Mat_{m\times n}(\mathbb{R})$ proizvodi ortogonalnu dekompoziciju od $\mathbb{R}^m$ i $\mathbb{R}^n$ u smislu da

$$\mathbb{R}^m = im(A) \oplus im(A)^{\perp} = im(A) \oplus ker(A^T),$$

i

$$\mathbb{R}^n = ker(A) \oplus ker(A)^{\perp} = ker(A) \oplus im(A^T).$$

Prvo trebamo pronaći baze za četri fundamentalna podprostora $ker(A)$, $im(A)$, $ker(A^T)$, $im(A^T)$.

Prisjetimo se

- osnovne kolone iz A formiraju bazu za $im(A)$
- nenula redovi od U formiraju bazu za $im(A^T)$
- linearno nezavisni vektori iz opšteg rješenja sistema $Ax=0$ formiraju bazu za $ker(A)$.
- zadnjih $m-r$ redova iz P formira bazu za $ker(A^T)$

gdje je P nesingularna matrica takva da $PA=U$, a matrica U je u red ešelon obliku.

$$A=\begin{pmatrix}2&1&1\\-1&-1&0\\-2&-1&-1\end{pmatrix}\underset{}{\overset{I_v\leftrightarrow II_v}{\sim}}\begin{pmatrix}-1&-1&0\\2&1&1\\-2&-1&-1\end{pmatrix}\overset{III_v+II_v}{\sim}\begin{pmatrix}-1&-1&0\\2&1&1\\0&0&0\end{pmatrix}\overset{II_v+I_v\cdot 2}{\sim}$$

$$\sim\begin{pmatrix}-1&-1&0\\0&-1&1\\0&0&0\end{pmatrix}\underset{II_v\cdot(-1)}{\overset{I_v\cdot(-1)}{\sim}}\begin{pmatrix}1&1&0\\0&1&-1\\0&0&0\end{pmatrix}\overset{I_v-II_v}{\sim}\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}=E_A$$

$$im(A)=span\left\{\begin{pmatrix}2\\-1\\-2\end{pmatrix},\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right\},\quad im(A^T)=span\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$$

$$ker(A)=\{x\mid Ax=0\}$$

$$\bar{A}=[A\,|\,0]=\left[\begin{array}{ccc|c}2&1&1&0\\-1&-1&0&0\\-2&-1&-1&0\end{array}\right]\overset{III_v+I_v}{\sim}\left[\begin{array}{ccc|c}2&1&1&0\\-1&-1&0&0\\0&0&0&0\end{array}\right]\overset{I_v+II_v}{\sim}$$

$$\sim\left[\begin{array}{ccc|c}1&0&1&0\\-1&-1&0&0\\0&0&0&0\end{array}\right]\overset{II_v+I_v}{\sim}\left[\begin{array}{ccc|c}1&0&1&0\\0&-1&1&0\\0&0&0&0\end{array}\right]\Rightarrow rang\,A=rang\,\bar{A}<3$$

$$x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\quad x_3=t,\quad \begin{matrix}-x_2+x_3=0\\x_1+x_3=0\end{matrix}\quad\Rightarrow\quad\begin{matrix}x_2=t\\x_1=-t\end{matrix}$$

$$x=\begin{pmatrix}-t\\t\\t\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}t,\ t\in\mathbb{R}\qquad ker(A)=span\left\{\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$$

Ostalo je još da odredimo bazu za $ker(A^T)$.

$$\Big(\underbrace{\begin{matrix}2&1&1\\-1&-1&0\\-2&-1&-1\end{matrix}}_{=A}\Big|\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{=I}\Big)\overset{III_v+I_v}{\sim}\left(\begin{array}{ccc|ccc}2&1&1&1&0&0\\-1&-1&0&0&1&0\\0&0&0&1&0&1\end{array}\right)\overset{I_v+II_v}{\sim}\left(\begin{array}{ccc|ccc}1&0&1&1&1&0\\-1&-1&0&0&1&0\\0&0&0&1&0&1\end{array}\right)$$

$$\overset{II_v+I_v}{\sim}\left(\begin{array}{ccc|ccc}1&0&1&1&1&0\\0&-1&1&0&1&0\\0&0&0&1&0&1\end{array}\right)\overset{II_v\cdot(-1)}{\sim}\Big(\underbrace{\begin{matrix}1&0&1\\0&1&-1\\0&0&0\end{matrix}}_{=E_A}\Big|\underbrace{\begin{matrix}1&1&0\\0&-1&0\\1&0&1\end{matrix}}_{=P}\Big)\Rightarrow P=\begin{pmatrix}1&1&0\\0&-1&0\\1&0&1\end{pmatrix}.$$

Prema tome $\ker(A^T) = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$.

Dobili smo da je

$$\text{im}(A) = \text{span}\left\{\begin{pmatrix}2\\-1\\-2\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix}\right\}$$

$$\text{im}(A^T) = \text{span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$$

$$\ker(A) = \text{span}\left\{\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$$

$$\ker(A^T) = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$$

$$(2\ \ -1\ \ -2)\begin{pmatrix}1\\0\\1\end{pmatrix} = 2-2=0, \qquad (1\ \ -1\ \ -1)\begin{pmatrix}1\\0\\1\end{pmatrix} = 1-1=0$$

Kako je svaki vektor u bazi za $\text{im}(A)$ ortogonalan na svaki vektor u bazi za $\ker(A^T)$, slijedi da je $\text{im}(A) \perp \ker(A^T)$. Ista logika također objašnjava zašto $\ker(A) \perp \text{im}(A^T)$. Primjetimo da je $\text{im}(A)$ ravan kroz koordinatni početak u $\mathbb{R}^3$, a $\ker(A^T)$ je prava kroz koordinatni početak okomita na ovu ravan, pa iz zakona paralelograma slijedi

$$\text{im}(A) \oplus \ker(A^T) = \mathbb{R}^3$$

Slično, $\ker(A)$ je prava kroz koordinatni početak normalna na ravan koja je definisana sa $\text{im}(A^T)$, pa je

$$\ker(A) \oplus \text{im}(A^T) = \mathbb{R}^3.$$

(#) Za unitarni prostor V, šta je $V^{\perp}$? Šta je $\mathbf{0}^{\perp}$?

Rj: Prema definiciji $\underline{M^{\perp} = \{x \in V \mid \langle m, x \rangle = 0 \text{ za } \forall m \in M\}}$

$$V^{\perp} = \{x \in V \mid \langle m, x \rangle = 0 \text{ za } \forall m \in V\}$$

drugim riječima tražimo vektor iz V koji je okomit na sve vektore iz V

$$V^{\perp} = \mathbf{0}$$

$$\mathbf{0}^{\perp} = \{x \in V \mid \langle m, x \rangle = 0 \text{ za } \forall m \in \{\mathbf{0}\}\}$$

$$\Rightarrow \quad \mathbf{0}^{\perp} = V$$

Pronaći bazu za ortogonalni komplement od

$$\mathcal{M} = \text{span}\left\{\begin{pmatrix}1\\2\\0\\3\end{pmatrix}, \begin{pmatrix}2\\4\\1\\6\end{pmatrix}\right\}.$$

Rj. Znamo da: Za svaku matricu $A \in Mat_{m\times n}(\mathbb{R})$

$$\operatorname{im}(A)^{\perp} = \ker(A^T) \quad i \quad \ker(A)^{\perp} = \operatorname{im}(A^T)$$

Ako je $A = \begin{pmatrix}1 & 2\\2 & 4\\0 & 1\\3 & 6\end{pmatrix}$ tada je $\operatorname{im}(A) = \mathcal{M}$.

$\operatorname{im}(A^T)$ je $\ker(A^T)$

Znamo da zadnjih $m-r$ redova iz P formira bazu za $\ker(A^T)$ gdje je P nesingularna matrica takva da $PA = U$, a matrica U je u red ešelon obliku.

$$\left[\begin{array}{cc|cccc}\multicolumn{2}{c|}{\overbrace{\quad}^{A}} & \multicolumn{4}{c}{\overbrace{\qquad\qquad}^{I}}\\ 1 & 2 & 1 & 0 & 0 & 0\\ 2 & 4 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 3 & 6 & 0 & 0 & 0 & 1\end{array}\right] \underset{IV_v + I_v\cdot(-3)}{\overset{II_v + I_v\cdot(-2)}{\sim}} \left[\begin{array}{cc|cccc}1 & 2 & 1 & 0 & 0 & 0\\ 0 & 0 & -2 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & -3 & 0 & 0 & 1\end{array}\right] \underset{I_v + III_v\cdot(-2)}{\overset{II_v + III_v}{\sim}}$$

$$\sim \left[\begin{array}{cc|cccc}1 & 0 & 1 & 0 & & 0\\ 0 & 1 & -2 & 1 & 1 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & -3 & 0 & 0 & 1\end{array}\right] \overset{III_v - II_v}{\sim} \left[\begin{array}{cc|cccc}1 & 0 & 1 & 0 & -2 & 0\\ 0 & 1 & -2 & 1 & 1 & 0\\ 0 & 0 & 2 & -1 & 0 & 0\\ 0 & 0 & -3 & 0 & 0 & 1\\ \multicolumn{2}{c|}{\underbrace{\quad}_{E_A}} & \multicolumn{4}{c}{\underbrace{\qquad\qquad}_{P}}\end{array}\right]$$

$$\Rightarrow \mathcal{M}^{\perp} = \text{span}\left\{\begin{pmatrix}2\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\0\\1\end{pmatrix}\right\}$$

traženo rješenje

(#) Za svaki unitarni prostor V, dokazati da ako je $M \subseteq V$, tada je $M^{\perp}$ podprostor od V.

Rj. Prema definiciji $M^{\perp} = \{ x \in V \mid \langle m, x \rangle = 0 \text{ za } \forall m \in M \}$.

Trebamo provjeriti da li je $M^{\perp}$ zatvoreno u odnosu na vektorsko sabiranje i skalarno množenje.

Ako je $x, y \in M^{\perp} \Rightarrow \langle m, x \rangle = 0 = \langle m, y \rangle$ za $\forall m \in M$

$\Rightarrow \langle m, x+y \rangle = 0 \quad \forall m \in M \Rightarrow x+y \in M^{\perp}$.

Slično, za $\forall \alpha$ (α - skalar) $\langle m, \alpha x \rangle = \alpha \langle m, x \rangle = 0 \quad \forall m$

$\Rightarrow \alpha x \in M^{\perp}$.

$M^{\perp}$ jest podprostor od V

(#) Ako su $\mathcal{M}$ i $\mathcal{N}$ podprostori od n-dimenzionalnog unitarnog prostora, dokazati da su sljedeće tvrdnje tačne,

(a) $\mathcal{M} \subseteq \mathcal{N} \Rightarrow \mathcal{N}^\perp \subseteq \mathcal{M}^\perp$

(b) $(\mathcal{M}+\mathcal{N})^\perp = \mathcal{M}^\perp \cap \mathcal{N}^\perp$

(c) $(\mathcal{M} \cap \mathcal{N})^\perp = \mathcal{M}^\perp + \mathcal{N}^\perp$

Rj. a) Neka je $\mathcal{M} \subseteq \mathcal{N}$

Izaberimo proizvoljno $x \in \mathcal{N}^\perp$. Tada

$$x \perp \mathcal{N} \supseteq \mathcal{M} \Rightarrow x \perp \mathcal{M} \Rightarrow x \in \mathcal{M}^\perp$$

Tj. $\mathcal{N}^\perp \subseteq \mathcal{M}^\perp$.

b) Jednostavno primjetimo da je

$$x \in (\mathcal{M}+\mathcal{N})^\perp \Leftrightarrow x \perp (\mathcal{M}+\mathcal{N}) = \{m+n \mid m \in \mathcal{M} \text{ i } n \in \mathcal{N}\}$$

$$\Leftrightarrow x \perp \mathcal{M} \text{ i } x \perp \mathcal{N}$$

$$\Leftrightarrow x \in (\mathcal{M}^\perp \cap \mathcal{N}^\perp)$$

c) Znamo da je $\mathcal{M}^{\perp\perp} = \mathcal{M}$.

Sad ako iskoristimo dio (b) imamo

$$(\mathcal{M}^\perp + \mathcal{N}^\perp)^\perp = \mathcal{M}^{\perp\perp} \cap \mathcal{N}^{\perp\perp} = \mathcal{M} \cap \mathcal{N}$$

$$\Rightarrow (\mathcal{M} \cap \mathcal{N})^\perp = \mathcal{M}^\perp + \mathcal{N}^\perp$$

Izračunati URV faktorizaciju matrice

$$A = \begin{pmatrix} -4 & -2 & -4 & -2 \\ 2 & -2 & 2 & 1 \\ -4 & 1 & -4 & -2 \end{pmatrix}$$

koristeći elementarne red operacije zajedno sa Gram-Schmidt-ovim procesom ortogonalizacije.

Rj.: URV faktorizacija

Za svaku matricu $A \in Mat_{m\times n}(\mathbb{R})$ ranga r, postoji ortogonalna matrica $U_{m\times m}$ i $V_{n\times n}$ i nesingularna matrica $C_{r\times r}$ takva da

$$A = URV^T = U\begin{pmatrix} C_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}_{m\times n} V^T$$

- prvih r kolona u U čine ortonormiranu bazu za $im(A)$
- zadnjih $m-r$ kolona od U su ortonormirana baza za $ker(A^T)$
- prvih r kolona u V su ortonormirana baza za $im(A^T)$
- zadnjih $n-r$ kolona od V je ortonormirana baza za $ker(A)$.

Izračunajmo E_A

$$A = \begin{pmatrix} -4 & -2 & -4 & -2 \\ 2 & -2 & 2 & 1 \\ -4 & 1 & -4 & -2 \end{pmatrix} \underset{III_v:(-2)}{\overset{I_v:(-4)}{\sim}} \begin{pmatrix} 1 & 1/2 & 1 & 1/2 \\ 2 & -2 & 2 & 1 \\ 2 & -1/2 & 2 & 1 \end{pmatrix} \underset{III_v+I_v\cdot(-1)}{\overset{II_v+I_v\cdot(-2)}{\sim}}$$

$$\sim \begin{pmatrix} 1 & 1/2 & 1 & 1/2 \\ 0 & -3 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \end{pmatrix} \underset{\substack{I_v+II_v\cdot\left(\frac{1}{6}\right) \\ II_v:(-3)}}{\overset{III_v+II_v\cdot\left(\frac{1}{2}\right)}{\sim}} \begin{pmatrix} 1 & 0 & 1 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E_A$$

$$\Rightarrow \quad im(A) = \left\{ \begin{pmatrix} -4 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}, \quad im(A^T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$Ax = 0 \iff E_A x = 0$

$$\begin{aligned} x_1 + \quad + x_3 + \tfrac{1}{2}x_4 &= 0 \\ x_2 \quad &= 0 \end{aligned}$$

$x_3 = s, \ x_4 = t$

$x_1 = -s - \frac{1}{2}t \qquad \Rightarrow \qquad x = \begin{pmatrix} -s-\frac{1}{2}t \\ 0 \\ s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} t$

$x_2 = 0$

$$\ker(A) = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|ccc} -4 & -2 & -4 & -2 & 1 & 0 & 0 \\ 2 & -2 & 2 & 1 & 0 & 1 & 0 \\ -4 & 1 & -4 & -2 & 0 & 0 & 1 \end{array}\right) \quad \left(\underbrace{}_{A} \ \underbrace{}_{I}\right) \overset{III_v - I_v}{\underset{\substack{II_v : 2 \\ I_v : (-4)}}{\sim}} \left(\begin{array}{cccc|ccc} 1 & 1/2 & 1 & 1/2 & -1/4 & 0 & 0 \\ 1 & -1 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 3 & 0 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{II_v - I_v}{\sim} \left(\begin{array}{cccc|ccc} 1 & 1/2 & 1 & 1/2 & -1/4 & 0 & 0 \\ 0 & -3/2 & 0 & 0 & 1/4 & 1/2 & 0 \\ 0 & 3 & 0 & 0 & -1 & 0 & 1 \end{array}\right) \overset{III_v + II_v \cdot 2}{\sim} \left(\begin{array}{cccc|ccc} 1 & 1/2 & 1 & 1/2 & -1/4 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 1 \end{array}\right)$$

$$\Rightarrow \quad \ker(A^T) = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

Sad nije teško primijeniti Gram-Schmidtovu proces i dobiti ortonormirane baze za četri fundamentalna podprostora

$$\mathcal{B}_{im(A)} = \left\{ \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{B}_{ker(A^T)} = \left\{ \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

$$\mathcal{B}_{im(A^T)} = \left\{ \frac{1}{3}\begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{B}_{ker(A)} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{3\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ -1 \\ 4 \end{pmatrix} \right\}$$

(za vježbu ovaj dio detaljno raspisati)

Sad matrice U i V nije teško izračunati

$$U=\left(\mathcal{B}_{im(A)}\cup\mathcal{B}_{ker(A^T)}\right)=\frac{1}{3}\begin{pmatrix}-2 & -2 & -1\\ 1 & -2 & 2\\ -2 & 1 & 2\end{pmatrix}$$

i

$$V=\left(\mathcal{B}_{im(A^T)}\cup\mathcal{B}_{ker(A)}\right)=\frac{1}{3}\begin{pmatrix}2 & 0 & -3/\sqrt{2} & -1/\sqrt{2}\\ 0 & 3 & 0 & 0\\ 2 & 0 & 3/\sqrt{2} & -1/\sqrt{2}\\ 1 & 0 & 0 & 4/\sqrt{2}\end{pmatrix}.$$

Direktnim množenjem dobijemo

$$R=U^TAV=\begin{pmatrix}9 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

(#) Objasniti zašto je slika plus jezgro teorema:

$$\dim \operatorname{im}(A) + \dim \ker(A) = n \quad \text{za sve } m\times n \text{ matrice}$$

posljedica teoreme ortogonalne dekompozicije.

Rj.

Teorema ortogonalne dekompozicije

Za svaku matricu $A \in Mat_{m\times n}(\mathbb{R})$

$$\operatorname{im}(A)^{\perp} = \ker(A^T) \quad ; \quad \ker(A)^{\perp} = \operatorname{im}(A^T)$$

i vrijedi

$$\mathbb{R}^m = \operatorname{im}(A) \oplus \ker(A^T) \quad ; \quad \mathbb{R}^n = \ker(A) \oplus \operatorname{im}(A^T)$$

Od ranije znamo da

$$\operatorname{rang}(A) = r \iff \dim \operatorname{im}(A) = r$$

Sad primjetimo

$$\dim \operatorname{im}(A^T) = \operatorname{rang}(A^T) = \operatorname{rang}(A) = \dim \operatorname{im}(A)$$

$$n = \dim \mathbb{R}^n = \dim(\ker(A) \oplus \operatorname{im}(A^T)) =$$

$$= \dim \ker(A) + \dim \operatorname{im}(A^T) =$$

$$= \dim \ker(A) + \dim \operatorname{im}(A)$$

$$\Rightarrow \quad n = \dim \operatorname{im}(A) + \dim \ker(A)$$

q.e.d.

(#) Ako je $\mathcal{M}$ podprostor konačno-dimenzionalnog unitarnog prostora V, tada $V = \mathcal{M} \oplus \mathcal{M}^\perp$. Dokazati.

Rj.

Prvo pokažimo da je $\mathcal{M} \cap \mathcal{M}^\perp = 0$.

Izaberimo proizvoljno $x \in \mathcal{M} \cap \mathcal{M}^\perp \Rightarrow x \in \mathcal{M}$ i $x \in \mathcal{M}^\perp$

$\Rightarrow \langle x, x \rangle = 0$ (x je ortogonalan sam na sebe)

$\Rightarrow x = 0$ Kako je x proizvodjan to $\mathcal{M} \cap \mathcal{M}^\perp = 0$

Sad pokažimo da je $\mathcal{M} \oplus \mathcal{M}^\perp = V$.

Neka su $\mathcal{B}_\mathcal{M}$ i $\mathcal{B}_{\mathcal{M}^\perp}$ ortonormirane baze za $\mathcal{M}$ i $\mathcal{M}^\perp$ redom. Kako su $\mathcal{M}$ i $\mathcal{M}^\perp$ disjunktni, $\mathcal{B}_\mathcal{M} \cup \mathcal{B}_{\mathcal{M}^\perp}$ je ortonormirana baza za neki podprostor $\mathcal{S} = \mathcal{M} \oplus \mathcal{M}^\perp \subseteq V$. Ako je $\mathcal{S} \neq V$ znamo da $\mathcal{B}_\mathcal{M} \cup \mathcal{B}_{\mathcal{M}^\perp}$ možemo proširiti do baze za V. Sad ako iskoristimo Gram-Schmidtov proces ortogonalizacije možemo formirati ortonormiran skup vektora $\mathcal{E}$ takav da je $\mathcal{B}_\mathcal{M} \cup \mathcal{B}_{\mathcal{M}^\perp} \cup \mathcal{E}$ ortonormirana baza za V. Sad imamo

$\mathcal{E} \perp \mathcal{B}_\mathcal{M} \Rightarrow \mathcal{E} \perp \mathcal{M} \Rightarrow \mathcal{E} \subseteq \mathcal{M}^\perp \Rightarrow \mathcal{E} \subseteq \operatorname{span}(\mathcal{B}_{\mathcal{M}^\perp})$

(svaki vektor iz $\mathcal{E}$ je ortogonalan na svaki vektor iz $\mathcal{B}_\mathcal{M}$)

#kontradikcija

(zato što je $\mathcal{B}_\mathcal{M} \cup \mathcal{B}_{\mathcal{M}^\perp} \cup \mathcal{E}$ linearno nezavisan skup)

Prema tome $\mathcal{E}$ je prazan skup $\Rightarrow V = \mathcal{M} \oplus \mathcal{M}^\perp$.

q.e.d.

(#) Ako je $\mathcal{N}$ podprostor takav da $\mathcal{V} = \mathcal{M} \oplus \mathcal{N}$ i $\mathcal{N} \perp \mathcal{M}$ (svaki vektor u $\mathcal{N}$ je ortogonalan na svaki vektor u $\mathcal{M}$) tada $\mathcal{N} = \mathcal{M}^\perp$. Dokazati.

Rj.

Kako je $\mathcal{N} \perp \mathcal{M}$ to je $\mathcal{N} \subseteq \mathcal{M}^\perp$.

Znamo da, ako su $\mathcal{X}$ i $\mathcal{Y}$ podprostori vektorskog prostora $\mathcal{V}$, tada $\dim(\mathcal{X}+\mathcal{Y}) = \dim \mathcal{X} + \dim \mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y})$. ...(*)

Znamo da

$$\left.\begin{aligned} \mathcal{V} = \mathcal{M} \oplus \mathcal{M}^\perp &\overset{(*)}{\Rightarrow} \dim(\mathcal{V}) = \dim(\mathcal{M}) + \dim(\mathcal{M}^\perp) \\ \mathcal{V} = \mathcal{M} \oplus \mathcal{N} &\overset{(*)}{\Rightarrow} \dim(\mathcal{V}) = \dim(\mathcal{M}) + \dim(\mathcal{N}) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \dim \mathcal{N} = \dim \mathcal{M}^\perp$$

Kako je

$$\mathcal{N} \subseteq \mathcal{M}^\perp \text{ i } \dim \mathcal{N} = \dim \mathcal{M}^\perp \Rightarrow \mathcal{N} = \mathcal{M}^\perp.$$

q.e.d.

(#) Neka je $U_{m\times m} = (U_1 \mid U_2)$ particionisana ortogonalna matrica. Objasniti zašto $im(U_1)$ i $im(U_2)$ moraju biti ortogonalni komplementi.

Rj.

Znamo da

$$\underline{V = \mathcal{X} \oplus \mathcal{Y} \iff \forall v \in V \ \exists!\, x \in \mathcal{X}, y \in \mathcal{Y} \quad v = x + y} \iff \underline{\begin{matrix} \mathcal{B}_{\mathcal{X}} \cap \mathcal{B}_{\mathcal{Y}} = \emptyset \\ \mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\mathcal{Y}} \text{ baza za } V \end{matrix}}$$

... (*)

Kako su kolone matrice U međusobno ortogonalne to su kolone od U_1 baza za $im(U_1)$, a kolone od U_2 baza za $im(U_2)$ pa prema (*) imamo

$$\mathbb{R}^m = im(U_1) \oplus im(U_2)$$

i primjetimo da je $im(U_1) \perp im(U_2)$.

Znamo da, <u>ako je N podprostor takav da $V = M \oplus N$ i $N \perp M$ tada $N = M^{\perp}$.</u>

... (**)

Sad na osnovu (**) primjetimo da je $im(U_2) = im(U_1)^{\perp}$ tj. $im(U_1)$ i $im(U_2)$ su ortogonalni komplementi.

(#) Ako je $\mathcal{M}$ podprostor od n-dimenzionalnog unitarnog prostora $\mathcal{V}$, tada su sljedeće tvrdnje tačne.

(i) $\dim \mathcal{M}^{\perp} = n - \dim \mathcal{M}$;

(ii) $\mathcal{M}^{\perp\perp} = \mathcal{M}$. Dokazati.

Rj.

(i) Znamo da

Ako su $\mathcal{X}$ i $\mathcal{Y}$ podprostori vektorskog prostora $\mathcal{V}$, tada

$$\dim(\mathcal{X}+\mathcal{Y}) = \dim \mathcal{X} + \dim \mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y}).$$

$\mathcal{M}$ i $\mathcal{M}^{\perp}$ su komplementarni podprostori $\Rightarrow$

$$\Rightarrow \mathcal{B}_{\mathcal{M}} \cap \mathcal{B}_{\mathcal{M}^{\perp}} = \emptyset \text{ i } \mathcal{B}_{\mathcal{M}} \cup \mathcal{B}_{\mathcal{M}^{\perp}} \text{ je baza za } \mathcal{V}$$

$$\Rightarrow \quad \dim(\mathcal{V}) = \dim(\mathcal{M}) + \dim(\mathcal{M}^{\perp})$$

$$\Rightarrow \quad \dim(\mathcal{M}^{\perp}) = n - \dim(\mathcal{M})$$

q.e.d.

(ii) Prvo pokažimo da je $\mathcal{M}^{\perp\perp} \subseteq \mathcal{M}$. Ako je $x \in \mathcal{M}^{\perp\perp}$ proizvoljno tada $\mathcal{V} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ povlači $x = m + n$, gdje $m \in \mathcal{M}$ i $n \in \mathcal{M}^{\perp}$, pa

$$0 = \langle n, x\rangle = \langle n, m+n\rangle = \langle n, m\rangle + \langle n, n\rangle = \langle n, n\rangle \Rightarrow n = 0 \Rightarrow$$

$\Rightarrow x \in \mathcal{M}$, pa prema tome $\mathcal{M}^{\perp\perp} \subseteq \mathcal{M}$.

Na osnovu (i) znamo da $\dim \mathcal{M}^{\perp} = n - \dim \mathcal{M}$; $\dim \mathcal{M}^{\perp\perp} = n - \dim \mathcal{M}^{\perp}$ pa je $\dim \mathcal{M}^{\perp\perp} = \dim \mathcal{M}$, a odavde vidimo da je

$$\mathcal{M}^{\perp\perp} = \mathcal{M}$$

q.e.d.

Za svaku matricu $A \in Mat_{m\times n}(\mathbb{R})$ pokazati da je

$$im(A)^{\perp} = ker(A^T) \quad i \quad ker(A)^{\perp} = im(A^T)$$

Rj.

Posmatrajmo prostor $im(A)^{\perp}$. Za $\forall y \in \mathbb{R}^n$

$$x \in im(A)^{\perp} \iff \langle \underbrace{Ay}_{\in im(A)}, \underbrace{x}_{\in im(A)^{\perp}} \rangle = 0 \iff (Ay)^T x = 0$$

$$\iff y^T A^T x = 0 \iff \langle y, A^T x \rangle = 0$$

Primjetimo se

Ako je $\langle x, y \rangle = 0$ za $\forall x \in V$ tada $y = 0$.

Zašto?

$$\langle x, y \rangle = 0 \ \forall x \in V \Rightarrow \langle y, y \rangle = 0 \Rightarrow y = 0$$

$$\iff A^T x = 0 \iff x \in ker(A^T)$$

Prema tome $im(A)^{\perp} = ker(A^T)$,

q.e.d.

Ako u dobijenoj jednakosti zamjenimo A sa A^T dobićemo

$$im(A^T)^{\perp} = ker(A) \quad /^{\perp}$$

$$ker(A)^{\perp} = im(A^T)$$

q.e.d.